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Generalized Position and Momentum Tsallis Entropies

Angsula Ghosh[1] and Puspitapallab Chaudhuri[1]

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We use the generalized Tsallis entropy $S_\omega(q = \epsilon(1 - \Sigma_{i=1}^{W} p_i^q)/(q - 1)$ to study the information measurement in position and momentum space for simple quantum mechanical systems. We consider here the hydrogen atom in three dimensions and the D-dimensional harmonic oscillator to calculate the position and momentum entropies analytically for ground and excited states which involve classical orthognal polynomials. In both the cases we verify the generalized entropic uncertainty relation and pseudoadditivity relation. We also study the effect of screening on the entropies. We compare the present results with the corresponding results of the Shannon formalism.

1. INTRODUCTION

Since the development of the theory of information by Shannon (1949) it has been widely used to study the properties of complex microscopic systems, including the foundations of quantum mechanics, and above all in the science of control and automation of dynamical processes. An information measure closely related to the concept of entropy in thermodynamics and was defined by Shannon (1949) as

$$S_\omega = -\int \omega(x) \ln \omega(x)\, dx \tag{1}$$

The quantal entropy S_ω is, briefly, the expected amount of information present in the the probability distribitution $\omega(x)$ in x space. Information entropies have shown to play an important role in the quantum mechanical description of physical systems. A possible generalization of the Boltzman–Gibbs–Shannon entropy proposed by Tsallis (1988) for nonextensive systems has drawn

[1] Instituto de Física, Universidade de São Paulo, Caixa Postal 66318, 05315-970 São Paulo, Brazil.

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considerable attention in the physics community. Intensive research has aimed at formulating a generalized statistical mechanics, namely a Tsallis statistics and thermodynamics, which could be universal. Tsallis statistics has been successfully used to study a variety of problems in many different areas, such as the specific heat of hydrogen (Lucena *et al.*, 1995), the rigid rotator (Curilef and Tsallis, 1995), the infinite-range Ising ferromagnet. (Nobre and Tsallis, 1995), fractal random walks (Alemany and Zanette, 1994), q-quantum mechanics (Tsallis, 1994), turbulence in an electron plasma (Boghosian, 1996), the solar neutrino problem (Kaniadakis *et al.*, 1996), nonlinear dynamical systems (Tsallis *et al.* 1994), cosmology (Hamity and Barraco, 1996), and many other systems. The generalized nonextensive Tsallis entropy is defined as

$$S_{\omega}(q) = \epsilon \frac{1 - \Sigma_{i=1}^{W} p_i^q}{q - 1} \tag{2}$$

where ϵ is a conventional positive constant (we will consider $\epsilon = 1$), q is any real number that characterizes a particular statistics, and $\{p_i\}$ is a normalized probability distribution, $\Sigma_{i=1}^{W} p_i = 1$. In the limit of $q \to 1$ Eq. (2) yields the conventional Boltzmann–Shannon logarithmic expression (1) for the entropy. When $q = 1$, the physics is an extensive one. In all other cases we are led into the realm of nonextensivity. Various properties of the usual entropy have been proved to hold for the general one: positivity, equiprobability, concavity, and irrereversibility. Its connection with thermodynamics is now established and suitably generalizes the standard additivity as well as the Shannon theorem.

In quantum mechanics, problems are formulated either in coordinate (r) space or in momentum (k) space, depending on which is more convenient for the problem considered. Thus from (2) we can write

$$S_{\rho}(q) = \frac{1}{q - 1} (1 - \int d^D\mathbf{r}\ |\rho(\mathbf{r})|^q) \tag{3}$$

and

$$S_{\gamma}(p) = \frac{1}{p - 1} (1 - \int d^D\mathbf{k}\ |\gamma(\mathbf{k})|^p) \tag{4}$$

for the position and momentum space entropies, respectively, where $\rho(\mathbf{r}) = |\Psi(\mathbf{r})|^2$ and $\gamma(\mathbf{k}) = |\tilde{\Psi}(\mathbf{k})|^2$ are the quantum mechanical probability densities in r space (coordinate space) and k space (momentum space), respectively. Here $\tilde{\Psi}(\mathbf{k})$ stands for the Fourier transform of $\Psi(\mathbf{r})$, the r-space eigenfunction of a central potential,

$$\tilde{\Psi}(\mathbf{k}) = \frac{1}{(2\pi)^{n/2}} \int d^D\mathbf{r}\ \Psi(\mathbf{r}) e^{-i\mathbf{k},\mathbf{r}} \tag{5}$$

The two entropies $S_\rho(q)$ and $S_\gamma(p)$ can be combined in a suitable way to have new generalized uncertainty relations which are valid for a large class of systems, extensive or nonextensive. We can have an entropic uncertainty relation based on the Sobolev inequality (Massen and Uffink, 1988; Rajagopal, 1995). In this formulation the uncertainty relation becomes

$$\frac{[1 + (1 - p)S_\gamma(p)]^{1/2p}}{[1 + (1 - q)S_\rho(q)]^{1/2q}} \leq \left(\frac{\pi}{q}\right)^{-n/4q} \left(\frac{\pi}{p}\right)^{n/4p} \tag{6}$$

with the condition $1/p + 1/q = 2$.

For two independent systems A and B, the Tsallis entropy of the composed system A + B [so that the probability of the composed system is $p_{ij}(\mathrm{A} + \mathrm{B}) = p_i(\mathrm{A})p_j(\mathrm{B})$] satisfies the pseudoadditivity relation (Portesi and Plastino, 1996; Santos, 1997)

$$S_{\mathrm{A+B}}(q) = S_{\mathrm{A}}(q) + S_{\mathrm{A}}(q) + (1 - q)S_{\mathrm{A}}(q)S_B(q) \tag{7}$$

Here $S_{\mathrm{A+B}}(q)$ also represents the entropy of the joint probability state. The parameter q denotes the degree of the nonextensitivity of the system considered. For example, in case of interacting systems it implies the effect of long-range interactions. We observe that for $q \neq 1$ the entropies are nonadditive, i.e., $S_{\mathrm{A+B}}(q) \neq S_{\mathrm{A}}(q) + S_{\mathrm{B}}(q)$. Both of these relations (6) and (7) are fully characterized by the parameters p and q, which are arbitrary real numbers.

Recent years have witnessed a growing interest in the application of information entropies to the fundamental problems of quantum mechanics. Our objective in this work is to employ the generalized Tsallis entropy to study two fundamental physical systems: the isotropic harmonic oscillator with a potential of type $V_{\mathrm{HO}}(r) = \lambda^2 r^2/2$ and the hydrogen atom with Coulomb interaction. Yãnez *et al.* (1994) studied coordinate- and momentum-space entropies for the D-dimensional harmonic oscillator and the hydrogen atom using Shannon's formalism for both ground and excited states. They found simple analytical results for the ground states. But inordinate complications were encountered in treating the excited states due to the presence of polynomials in the excited-state wavefunctions. The logarithm of these polynomials presents difficulties in evaluating entropy integrals analytically. Recently Bhattacharya *et al.* (1998) presented a mathematical trick to deal with these complications for hydrogenic excited states. It is interesting to note that the Tsallis formalism not only provides a general entropic relation valid for both extensive and nonextensive systems, it also simplifies the calculational labor to a great extent. The Shannon entropy is a particular or limiting case of the Tsallis entropy (for $q \to 1$), so it can always be calculated from the latter.

We can evaluate almost all the entropy integrals analytically using standard integrals (Gradshetyn and Ryzhik, 1965). Our second objective is to examine the effect of screening on the generalized entropic relations (6) and (7) and thus gain some physical insight about S_ω. To the best of our knowledge there is no previous attempt to evaluate the information entropy integrals for these systems using the Tsallis generalized entropy.

The outline of the paper is as follows: In section 2 we consider the case of the D-dimensional harmonic oscillator and calculate the position and momentum information entropies analytically and also verify the relations (6) and (7). Relation (7) also allows us to obtain the expression for the joint probability state in all the cases. In Section 3 we do the same for the hydrogen atom in three dimensions. In Section 4 we investigate the effect of screening on $S_\rho(q)$ and $S_\gamma(p)$, using the Hulthén potential (Hulthén, 1942) as a model for the screened Coulomb interaction.

2. THE *D*-DIMENSIONAL HARMONIC OSCILLATOR

The normalized eigenfunctions of the D-dimensional harmonic oscillator in position and momentum space are given by

$$\Psi_{n,l,\{m\}}(\mathbf{r}) = \left[\frac{2n!\lambda^{l+D/2}}{\Gamma(n+l+D/2)}\right]^{1/2} r^l\, e^{-\lambda r^2/2} L_n^{l+D/2-1}(\lambda r^2) Y_{l,\{m\}}(\Omega_D) \tag{8}$$

and

$$\tilde{\Psi}_{n,l,\{m\}}(\mathbf{k}) = \left[\frac{2n!\lambda^{-l-D/2}}{\Gamma(n+l+D/2)}\right]^{1/2} k^l\, e^{-k^2/2\lambda} L_n^{l+D/2-1}\left(\frac{k^2}{\lambda}\right) Y_{l,\{m\}}(\tilde{\Omega}_D) \tag{9}$$

where the $Y_{l,\{m\}}(\Omega_D)$ are the hyperspherical harmonics (Yãnez *et al.*, 1994) and $\Omega_D(\tilde{\Omega}_D)$ is the solid angle in position (momentum) space. The symbol $L_n^\alpha(t)$ denotes the Laguerre polynomial.

The position-space information integral in the generalized form for the harmonic oscillator can be obtained as

$$S_\rho(q) = \frac{1}{q-1}\left[1 - \left(\frac{2n!\lambda^{l+D/2}}{\Gamma(n+l+D/2)}\right)^q I_1 I_2\right] \tag{10}$$

where the integrals I_1 and I_2 are given by

$$I_1 = \int r^{2lq+d-1}\, e^{-\lambda r^2 q}[L_n^{l+D/2-1}(\lambda r^2)]^{2q}\, dr \tag{11}$$

and

$$I_2 = \int [Y_{l,\{m\}}(\Omega_D)]^{2q}\, d\Omega_D \tag{12}$$

I_1 and I_2 are the generalized entropies for the Laguerre polynomial and hyperspherical harmonics, respectively. In momentum space we can write the generalized entropy as

$$S_\gamma(p) = \frac{1}{p-1}\left[1 - \left(\frac{2n!\lambda^{-l-D/2}}{\Gamma(n+l+D/2)}\right)^p I_3 I_2\right] \tag{13}$$

where

$$I_3 = \int k^{2lp+d-1}\, e^{-k^2 p/\lambda}[L_n^{l+D/2-1}(k^2/\lambda)]^{2p}\, dk \tag{14}$$

We give the explicit expressions for the entropies of the one-, two-, and three-dimensional harmonic oscillator for both the ground and excited states.

For the ground state, $n = 0$, we can write in position space

$$S_\rho^{\mathrm{HO}}(q) = \frac{1}{q-1}\left[1 - \left(\frac{\lambda}{\pi}\right)^{D(q-1)/2} \frac{1}{q^{D/2}}\right] \tag{15}$$

and in momentum space

$$S_\gamma^{\mathrm{HO}}(p) = \frac{1}{p-1}\left[1 - \left(\frac{1}{\lambda\pi}\right)^{D(p-1)/2} \frac{1}{p^{D/2}}\right] \tag{16}$$

where D is the dimension of the system. The entropy expressions in Eqs. (15) and (16) satisfy the uncertainty relation (6) and pseudoadditivity relation (7), which is independent of the potential strength of the harmonic oscillator, as expected, and also recover the results of Yãnez *et al.* (1994) in the limit $q \to 1$. We can also obtain the generalized joint entropy expression from (7) as

$$S_{\rho,\gamma}^{\mathrm{HO}}(q) = \frac{1}{q-1}\left[1 + \frac{\pi^{D(1-q)}}{q^D}\right] \tag{17}$$

Let us now consider the excited states of the harmonic oscillator, which needs some separate attention.

2.1. One Dimension

In the excited state $n = 1$ we get

$$S_\rho^{\mathrm{HO}}(q) = \frac{1}{q-1}\left[1 - \left(\frac{2}{\sqrt{\pi}}\right)^q \lambda^{(q-1)/2}\,\frac{\Gamma(q+1/2)}{q^{q+1/2}}\right] \tag{18}$$

and

$$S_\gamma^{\mathrm{HO}}(p) = \frac{1}{p-1}\left[1 - \left(\frac{2}{\sqrt{\pi}}\right)^p \left(\frac{1}{\lambda}\right)^{(p-1)/2}\,\frac{\Gamma(p+1/2)}{p^{p+1/2}}\right] \tag{19}$$

From the general expression of the information entropies (10) and (13) for an arbitrary value of n we plot S_ρ and S_γ for different states with quantum number n varying from 0 to 20 in Fig. 1 for $\lambda = 1/2$. Here we choose two

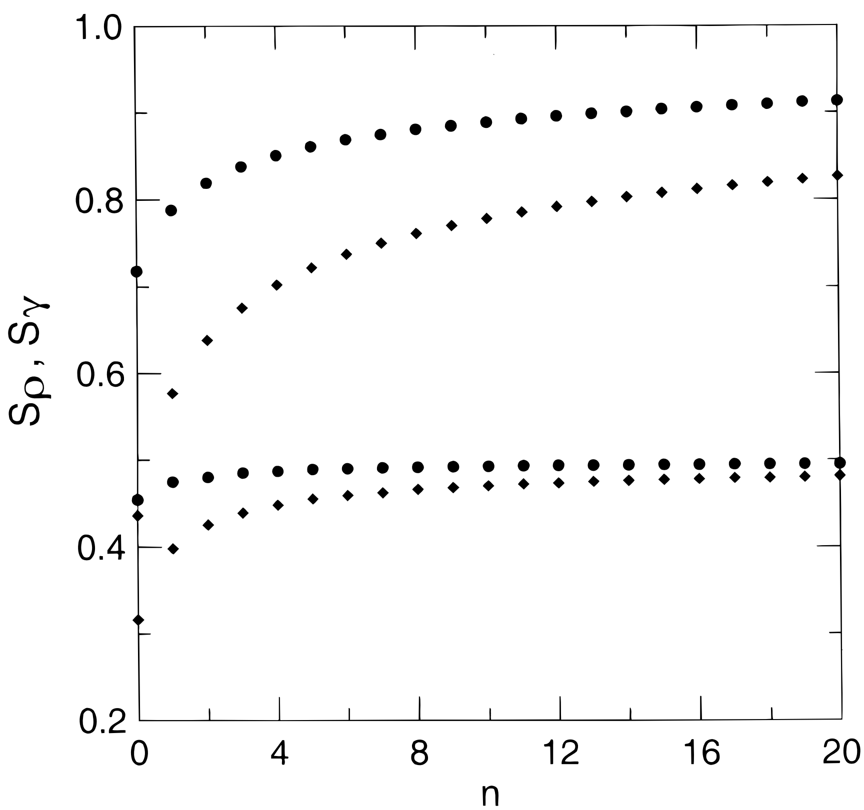

Fig. 1. Information entropies in position space S_ρ and momentum space S_γ of the one-dimensional harmonic oscillator with strength $\lambda = 1/2$ and $p = q = 2$, 3 versus principal quantum number n. The upper (lower) set of curves is for $p = q = 2$ (3).

particular values of p and q, $p = q = 2$ and $p = q = 3$, to exemplify the dependence of S_ρ and S_γ on them. We observe that both quantities initially increases with n and then tend to saturate for higher values of n. We also note that the entropies decrease with the increase of p and q for the same value of n. In this context we comment that for $p < 1$, $q < 1$ we observe an increase of entropy with increase of p and q. The maximum value of entropy occurs at $p = 1$, $q = 1$ (Portesi and Plastino, 1996).

2.2. Two Dimensions

For the first excited state, $n = 0$, $m = 1$, we get

$$S_\rho^{\mathrm{HO}}(q) = \frac{1}{q-1}\left[1 - \left(\frac{\lambda}{\pi}\right)^{q-1} \frac{\Gamma(q+1)}{q^{q+1}}\right] \tag{20}$$

and in momentum space,

$$S_\gamma^{\mathrm{HO}}(p) = \frac{1}{p-1}\left[1 - \left(\frac{1}{\lambda\pi}\right)^{p-1} \frac{\Gamma(p+1)}{p^{p+1}}\right] \tag{21}$$

The values of $S_\rho(q)$ and $S_\gamma(p)$ for the excited state with $n = 0$, $m = -1$ are same as those with $n = 0$, $m = 1$. As observed in one dimension, the entropies satisfy the relations (6) and (7) are independent of the potential strength λ. From Eq. (7) we find the joint entropy as

$$S_{\rho,\gamma}^{\mathrm{HO}}(\{p, q\}) = \frac{1}{q-1}\left[1 + \frac{1}{\pi^{2(q-1)}}\,[\Gamma(q+1)]^2 q^{2q+2}\right] \tag{22}$$

2.3. Three Dimensions

We can write for information entropy in position space with $n = 0$, $l = 1$, $m = 0$,

$$S_\rho^{\mathrm{HO}}(q) = \frac{1}{q-1}\left[1 - \frac{3^q}{(2\pi)^{q-1}} \frac{\lambda^{3(q-1)/2}}{(2q+1)q^{q+3/2}} \frac{\Gamma(q+3/2)}{[\Gamma(5/2)]^q}\right] \tag{23}$$

and in momentum space,

$$S_\gamma^{\mathrm{HO}}(p) = \frac{1}{p-1}\left[1 - \frac{3^p}{(2\pi)^{p-1}} \frac{\lambda^{-3(p-1)/2}}{(2p+1)p^{p+3/2}} \frac{\Gamma(p+3/2)}{[\Gamma(5/2)]^p}\right] \tag{24}$$

If we consider the case of excited states with $n = 0$, $l = 1$, $m = 1$, we get

$$S_{\rho}^{\text{HO}}(q) = \frac{1}{q-1}\left[1 - \frac{(3/8)^q}{\pi^{q-3/2}}\frac{\lambda^{3(q-1)/2}}{q^{q+3/2}}\frac{\Gamma(q+1)}{[\Gamma(5/2)]^q}\right] \tag{25}$$

and

$$S_{\gamma}^{\text{HO}}(p) = \frac{1}{p-1}\left[1 - \frac{(3/8)^p}{\pi^{p-3/2}}\frac{\lambda^{-3(p-1)/2}}{p^{p+3/2}}\frac{\Gamma(p+1)}{[\Gamma(5/2)]^p}\right] \tag{26}$$

The values of the $S_\rho(q)$ and $S_\gamma(p)$ are the same for the states (n, l, m) and $(n, l, -m)$. In three dimensions the combined entropies in (6) and (7) are seen to be independent of the potential strength of the harmonic oscillator, as has been observed before. From (7) we find

$$S_{\rho,\gamma}^{\text{HO}}(\{p, q\}) = \frac{1}{q-1}\left[1 + \frac{(3/8)^{2q}}{\pi^{2q-3}[\Gamma(5/2)]^{2q}}\frac{[\Gamma(q+1)]^2}{q^{2q+3}}\right] \quad \text{for} \quad m = \pm 1 \tag{27}$$

Moreover, we verify that in the limit $q \to 1$ the expressions for the information entropies as calculated in the above section, both in coordinate and momentum space, identically give the Shannon–Boltzmann entropy as obtained by Yãnez *et al.* (1994) for the same system. So we see that the Shannon entropy is just a particular case of the more generalized Tsallis entropy.

3. HYDROGEN ATOM

The wavefunction for the ground state of the hydrogen atom (in atomic units) is given by

$$\Psi_{1s}(\mathbf{r}) = \frac{1}{\sqrt{\pi}} e^{-r} \tag{28}$$

The corresponding wavefunction in momentum space is given by

$$\Psi_{1s}(\mathbf{k}) = \frac{1}{\pi}\frac{2\sqrt{2}}{(1+k^2)^2} \tag{29}$$

By applying similar procedures described in the previous section we get the information entropies $S_\rho(q)$ and $S_\gamma(p)$ in a closed analytical form:

$$S_{\rho}^{\text{H}}(q) = \frac{1}{q-1}\left[1 - \frac{1}{q^3\pi^{q-1}}\right] \tag{30}$$

$$S_{\gamma}^{\text{H}}(p) = \frac{1}{p-1}\left[1 - \frac{2^{3p+2}}{\pi^{2p-1}} I_4\right] \tag{31}$$

where

$$I_4 = \int_0^\infty \frac{k^2}{(1 + k^2)^{4p}}\, dk = \frac{\sqrt{\pi}\, \Gamma(4p - 3/2)}{4\Gamma(4p)} \tag{32}$$

In the limit of extensivity we can easily verify from Eq. (30) that

$$\lim_{q\to 1} S_\rho^{\mathrm{H}}(q) = 3 + \ln \pi \tag{33}$$

The rhs of the above equation is exactly the same expression as obtained earlier (Yanez *et al.*, 1994; Bhattacharya *et al.*, 1998) in the case of the hydrogen atom using the Shannon entropy formalism. Thus once more we observe through a simple physical system that the Tsallis entropy is a more generalized version of the conventional Boltzmann–Shannon expression. The entropy of the joint probabilty state in this case can be obtained in a straightforward manner from Eq. (7) using (30) and (31).

Now let us consider the excited states of the hydrogenic atom. The calculations for the excited states are not as trivial as in the ground state. However, we can obtain the results analytically with some algebraic manipulations. Here we consider H(2*s*) and H(2*p*) states. The H(2*s*) wavefunctions in position and momentum space are given by

$$\Psi_{2s}(\mathbf{r}) = \frac{1}{4\sqrt{2\pi}}\,(2 - r)\, e^{-r/2} \tag{34}$$

and

$$\tilde{\Psi}_{2s}(\mathbf{k}) = \frac{16}{\pi}\,\frac{1 - 4k^2}{(1 + 4k^2)^3} \tag{35}$$

respectively. For (34) the position-space entropy becomes

$$S_\rho^{\mathrm{H}}(q) = \frac{1}{q - 1}\left[1 - \frac{1}{2^{5q-2}\,\pi^{2q-1}}\, I_5\right] \tag{36}$$

where

$$I_5 = \int_0^\infty r^2 (2 - r)^{2q}\, e^{-rq}\, dr \tag{37}$$

With the change of variable $r - 2 = x$ we can rewrite and evaluate I_5 analytically as

$$I_5 = \int_{-2}^\infty (x + 2)^2\, x^{2q}\, e^{-(x+2)q}\, dx = A + B + C \tag{38}$$

with

$$A = 4e^{-2q}\left[\frac{\Gamma(2q+1)}{q^{2q+1}} + \frac{2^{2q+3}}{2q+1}\,{}_1F_1(1, 2q+2; -2q)\right]$$

$$B = 2e^{-2q}\left[\frac{\Gamma(2q+2)}{q^{2q+2}} - \frac{2^{2q+2}}{q+1}\,{}_1F_1(1, 2q+3; -2q)\right] \tag{39}$$

$$C = e^{-2q}\left[\frac{\Gamma(2q+3)}{q^{2q+3}} + \frac{2^{2q+3}}{2q+3}\,{}_1F_1(1, 2q+4; -2q)\right]$$

where ${}_1F_1[\cdot]$ are confluent hypergeometric functions.

Similarly for (35) the momentum-space entropy becomes

$$S_\gamma^{\mathrm{H}}(p) = \frac{1}{p-1}\left[1 - \frac{4^{4p+1}}{\pi^{2p-1}} I_6\right] \tag{40}$$

where

$$I_6 = \int_0^\infty \frac{(1-4k^2)^{2p}}{(1+4k^2)^{6p}}\, k^2\, dk \tag{41}$$

To evaluate I_6, we substitute $(1+4k^2)^{-1} = x$ and obtain the result in terms of hypergeometric ${}_2F_1[\cdot]$ functions. We thus obtain

$$\begin{aligned} I_6 = {} & \frac{\Gamma(2p+1)}{2^{4p+5/2}}\left[\frac{\Gamma(3/2-6p)}{\Gamma(5/2-4p)} + \frac{\Gamma(-3/2+4p)}{\Gamma(-1/2+6p)}\right] \\ & {}_2F_1\left(-\frac{1}{2}, 4p-\frac{3}{2}; 6p-\frac{1}{2}; \frac{1}{2}\right) \\ & + \sqrt{\pi}\, 2^{2p-6}\left[\frac{\Gamma(-3/2+6p)}{\Gamma(6p)}\right] {}_2F_1\left(1-6p, -2p; \frac{5}{2}-6p; \frac{1}{2}\right) \end{aligned} \tag{42}$$

Next, we consider the excited hydrogenic state with $n = 2$ and $l = 1$. The wavefunction for the H(2p) state in coordinate space is given by

$$\Psi_{2p}(\mathbf{r}) = \frac{1}{\sqrt{24}}\, re^{-r/2}\, Y_{1m}(\Omega) \tag{43}$$

where $m = 0, \pm 1$. The corresponding position-space entropies are obtained as

$$S_\rho^{\mathrm{H}}(q) = \frac{1}{q-1}\left[1 - \frac{2^{2-5q}\,\pi^{1-q}}{q^{2q+3}}\frac{\Gamma(2q+3)}{2q+1}\right] \quad \text{for} \quad m = 0 \tag{44}$$

$$S_\rho^{\mathrm{H}}(q) = \frac{1}{q-1}\left[1 - \frac{2^{1-6q}\pi^{3/2-q}}{q^{2q+3}}\frac{\Gamma(q+1)\Gamma(2q+3)}{\Gamma(q+3/2)}\right]$$

$$\text{for} \quad m = \pm 1 \tag{45}$$

The momentum-space wavefunction for H(2p) is given by

$$\tilde{\Psi}_{2p}(\mathbf{k}) = -i \frac{128}{\sqrt{3\pi}} \frac{k}{(1 + 4k^2)^3} Y_{1m}(\tilde{\Omega}) \tag{46}$$

The expressions for the momentum-space entropies become

$$S_\gamma^{\mathrm{H}}(p) = \frac{1}{p - 1}\left[1 - \frac{(-1)^p 4^{6p+1}}{\pi^{2p-1}(2p + 1)} I_7\right] \qquad \text{for} \quad m = 0 \tag{47}$$

$$S_\gamma^{\mathrm{H}}(p) = \frac{1}{p - 1}\left[1 - \frac{(-1)^p 2^{11p+1}}{\pi^{2p-3/2}} \frac{\Gamma(p + 1)}{\Gamma(p + 3/2)} I_7\right] \qquad \text{for} \quad m = \pm 1 \tag{48}$$

where

$$I_7 = \int_0^\infty \frac{k^{2p+2}}{(1 + 4k^2)^{6p}}\, dk = \left(\frac{1}{4}\right)^{p+2} \frac{\Gamma(2p + 2)\Gamma(4p - 2)}{\Gamma(6p)} \tag{49}$$

where we use the same substitution as in I_6. Here we comment that although the Shannon entropy is a particular case of the Tsallis entropy, the calculations for the excited states of hydrogen become much more complicated due to the presence of the log term and need a lot of algebraic manipulation (Bhattacharya *et al.*, 1998). However, within the Tsallis formalism we can evaluate the integrals in a straightforward way using gamma and hypergeometric functions. This can be considered as an advantage of using the Tsallis entropy. Both H(2s) and H(2p) satisfy the uncertainty relation and pseudoadditivity relation, as expected.

4. EFFECT OF SCREENING ON THE INFORMATION ENTROPY

The Yukawa potential is often used as a model for screened and cutoff Coulomb interactions, but the eigenvalue for this interaction cannot be solved analytically. We have therefore chosen to work with the two-parameter (V_0, a) Hulthén potential given by (Hulthén, 1942)

$$V(r) = -V_0 \frac{e^{-r/a}}{1 - e^{-r/a}} \qquad \text{where} \quad V_0 a > 0 \tag{50}$$

This potential behaves like a Coulomb potential $V_c = -V_0 a/r$ at small values of r, whereas for large values of r it decreases exponentially, so that its capacity for bound states is smaller than that of V_c. Alternatively, (50) will exhibit the same behavior as $a \to \infty$. If we work in atomic units, the correct Coulomb limit will be obtained as $a \to \infty$ and $V_0 a \to 1$. With a regarded as a screening parameter, the Hulthén potential has been widely used as a judicious model for screened interaction. In the following we make use of wavefunctions given in Flügge (1974) and Laha *et al.* (1988) to study the effect of screening on the position and momentum information entropies.

The normalized ground-state (1*s*) coordinate-space wavefunction for the Hulthén potential can be written as

$$\Psi_{1s}(\mathbf{r}) = \frac{1}{\sqrt{\pi}}\left(\frac{\alpha_1}{a}\right)^{3/2} e^{-(\alpha_1/a)r} \tag{51}$$

with

$$\alpha_1 = V_0 a^2 - \frac{1}{2} \tag{52}$$

The momentum-space wavefunction corresponding to (51) is given by

$$\tilde{\Psi}_{1s}(\mathbf{k}) = \frac{\alpha_1(2\alpha_1 a)^{3/2}}{\pi(\alpha_1^2 + a^2k^2)^2} \tag{53}$$

Using Eqs. (3), (4), (51), and (53), we get

$$S_\rho^{\mathrm{HP}}(q) = \frac{1}{q-1}\left[1 - \frac{\pi^{1-q}}{q^3}\left(\frac{\alpha_1}{a}\right)^{3(q-1)}\right] \tag{54}$$

and

$$S_\gamma^{\mathrm{HP}}(p) = \frac{1}{p-1}\left[1 - \left(\frac{\alpha_1}{a}\right)^{3(1-p)} \frac{2^{3p}}{\pi^{2p-3/2}} \frac{\Gamma(4p-3/2)}{\Gamma(4p)}\right] \tag{55}$$

We point out here that in the unscreening limit $\alpha_1/a \to 1$, we recover the results for the Coulomb potential as in H(1*s*) given in Eqs. (30) and (31). One can easily verify using Eqs. (54) and (55) that the uncertainty relation and pseudoadditivity relation are independent of the screening parameter α_1/a.

The exact 2*s* wavefunction for the Hulthén potential in coordinate space can be obtained similarly from the general *s*-state eigenfunction given in Flügge (1974). The normalized 2*s* wavefunction is given by

$$\Psi_{2s}(\mathbf{r}) = \frac{N_1}{\sqrt{8\pi}} e^{-(\alpha_1/a)r}(1 - \mathscr{C}e^{-r/a}) \tag{56}$$

where

$$N_1 = \left[\left(\frac{2\alpha_1}{a} \right)^{-3} - 2\mathscr{C} \left(\frac{2\alpha_1 + 1}{a} \right)^{-3} + \mathscr{C}^2 \left(\frac{2\alpha_1 + 2}{a} \right)^{-3} \right]^{-1/2} \tag{57}$$

with $\mathscr{C} = (2\alpha_1 + 3)/(2\alpha_1 + 1)$. The corresponding normalized $2p$ wavefunction as obtained by Laha *et al.* (1988) using a supersymmetry-inspired radial ladder operator is given by

$$\Psi_{2p}(\mathbf{r}) = \frac{N_2}{\sqrt{2}} e^{-(\alpha_2/a)r} (1 - e^{-r/a}) Y_{1m}(\Omega) \tag{58}$$

where

$$\alpha_2 = \frac{V_0 a^2}{2} - 1 \tag{59}$$

and

$$N_2 = \left[\left(\frac{2\alpha_2}{a} \right)^{-3} - 2 \left(\frac{2\alpha_2 + 1}{a} \right)^{-3} + \left(\frac{2\alpha_2 + 2}{a} \right)^{-3} \right]^{-1/2} \tag{60}$$

and $Y_{1m}(\Omega)$ is the scalar spherical harmonic with $m = 0, \pm 1$. Since the radial parts of the above two wavefunctions are almost the same in structure, the calculation of the radial part will be almost the same. However, the existence of the variable angular part in $2p$ makes it more generalized and so we consider only the $2p$ wavefunction here to calculate the information entropies. We have to consider two cases, depending on the value of m. The entropies become

$$S_\rho^{\mathrm{HP}}(q) = \frac{1}{q-1} \left[1 - \frac{(\sqrt{3}N_2)^{2q}}{2q+1} \frac{\pi^{1-q}}{2^{3q-2}} I_8 \right] \quad \text{for} \quad m = 0 \tag{61}$$

$$S_\rho^{\mathrm{HP}}(q) = \frac{1}{q-1} \left[1 - \frac{(\sqrt{3}N_2)^{2q} 2^{1-4q}}{\pi^{q-3/2}} I_8 \right] \quad \text{for} \quad m = \pm 1 \tag{62}$$

where

$$I_8 = \int_0^\infty e^{-(2q\alpha/a)r} (1 - e^{-r/a})^{2q} r^2 \, dr \tag{63}$$

Introducing the change of variable $x = e^{-r/a}$, we rewrite and evaluate I_8 as (Gradshetyn and Ryzhik, 1965)

$$I_8 = 2a^3 \int_0^1 (1 - x)^{2q} x^{2\alpha_2 q}{}_{-1} \ln x \, dx$$
$$= 2a^3 B(2\alpha_2 q, 2q + 1)[\psi(2\alpha_2 q) - \psi(2q(\alpha_2 + 1) + 1)] \qquad (64)$$

where $B(x, y)$ and $\psi(x)$ denote the usual beta and psi functions, respectively (Gradshetyn and Ryzhik, 1965). We note that in the case of the $2s$ wave function we will get a similar integral to I_8 with $\ln x$ replaced by $(\ln x - \ln \mathscr{C})$ due to the presence of the constant $\mathscr{C}$ in the wave function, which can alse be evaluated analytically.

In handling the angular part in all the above cases we used the following basic integrals:

$$\int_0^\pi \cos^{2q}(\theta) \sin(\theta) \, d\theta = \frac{2}{2q + 1} \qquad (65)$$

$$\int_0^\pi \sin^{2q+1}(\theta) \, d\theta = \frac{\pi^{1/2}\Gamma(q + 1)}{\Gamma(q + 3/2)} \qquad (66)$$

The momentum-space wavefunction for the $2p$ state is

$$\tilde{\Psi}_{2p}(\mathbf{k}) = -i \frac{2N_2}{\sqrt{\pi}} (2\alpha_2 + 1) a^4 k \times \frac{(\alpha_2^2 + a^2k^2) + [(\alpha_2 + 1)^2 + a^2k^2]}{\{(\alpha_2^2 + a^2k^2)[(\alpha_2 + 1)^2 + a^2k^2]\}^2} Y_{1m}(\tilde{\Omega}) \qquad (67)$$

Now using (67), we obtain the momentum-space information entropy as

$$S_\gamma^{\text{HP}}(p) = \frac{1}{p - 1}\left[1 - (-1)^p \frac{4\pi^{1-p}}{2p + 1} (\sqrt{3}N_2(2\alpha_2 + 1)a^4)^{2p} I_9\right] \quad \text{for} \quad m = 0 \qquad (68)$$

$$S_\gamma^{\text{HP}}(p) = \frac{1}{p - 1}\left[1 - \frac{(-1)^p \, 2^{1-p}}{\pi^{2p-3/2}} (\sqrt{3}N_2(2\alpha_2 + 1)a^4)^{2p} \frac{\Gamma(p + 1)}{\Gamma(p + 3/2)} I_9\right] \quad \text{for} \quad m = \pm 1 \qquad (69)$$

where

$$I_9 = \int_0^\infty \frac{(\alpha_2^2 + (\alpha_2 + 1)^2 + 2a^2k^2)^{2p} k^{2p+2}}{(\alpha_2^2 + a^2k^2)^{4p}((\alpha_2 + 1)^2 + a^2k^2)^{4p}} \, dk \qquad (70)$$

I_9 can be evaluated numerically for general values of p. However, for small values of p, analytical results are obtainable. For example, if we consider $p = 1/2$, we can evaluate I_9 as

$$I_9 = \frac{\ln(1 + \alpha_2) - \ln \alpha_2}{(1 + 2\alpha_2)a^4} \tag{71}$$

In conclusion, we have employed the Tsallis entropy formalism for simple quantum mechanical systems that are exactly solvable. We have calculated the position and momentum entropies for the D-dimensional harmonic oscillator (with $D = 1$, 2, and 3) for both the ground and excited states and observed the variation of S_ρ and S_γ with the principal quantum number n and the characteristic parameters p and q. We have done the same with the hydrogen atom in three dimensions. In both cases we have given analytical results for the entropy integrals that involve classical orthogonal polynomials like Laguerre and Hermite polynomials. We have also verified the generalized entropic relations. In the case of the harmonic oscillator they have been found to be independent of the potential strength. Finally, we have considered the screened Coulomb potential. On very general grounds one knows that in a screened hydrogenic system an electron experiences a more repulsive environment than is found in a pure Coulomb field. A screened Coulomb wavefunction is thus likely to be pushed apart leading to a relatively diffused probability density in position space. Consequently, S_ρ^{HP} should be greater than S_ρ^{H}. Our result in (54) clearly indicates this since for all real situations $\alpha_1/a < 1$. Understandably, the opposite will happen for $S_\gamma^{\mathrm{H}}P$. Moreover, the uncertainty relations have been found to be independent of the screening parameter, in other words, we have seen that the combination of enropies in (6) and (7) is identical both for the Coulomb and screened Coulomb interaction. This result is indeed physically appealing, since the net information content is invariant to the scale transformation.

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